Monetary Economics

Extension 1: The Sticky-Wages Extension

Olivier Loisel

ENSAE

October – November 2024

Motivation

- In the basic NK model (presented in Chapter 1), the **labor market** is assumed to be **perfectly competitive**:
  - all private agents are wage-takers, not wage-setters,
  - the nominal wage freely adjusts so as to clear the labor market.
- However, there is **empirical evidence** of nominal-wage stickiness, as seen in the general introduction.
- This extension introduces **nominal-wage stickiness** into the basic NK model and analyzes its implications for MP.
- Following Erceg et al. (2000), nominal-wage stickiness is modelled in the same way as price stickiness, by assuming that workers
  - have **monopoly power**, so that they are wage-setters, not wage-takers,
  - face **Calvo-type constraints** on the frequency with which they can adjust wages.

Main results

1. Wage-inflation and price-inflation dynamics are described by **similar equations** (closely related to the NK Phillips curve).

2. There are **four distortions**:
   - monopolistic competition and nominal rigidities in the goods market,
   - monopolistic competition and nominal rigidities in the labor market.

3. MP should have **three objectives**: stabilizing the output gap, price inflation, and wage inflation.

4. In a specific case, **optimal MP** fully stabilizes a weighted average of price- and wage-inflation.

Outline

Production function

- Each firm i has the same **production function** as in Chapter 1:

$$Y_t(i) = A_t N_t(i)^{1-\alpha}.$$

- However, $N_t(i)$ is now an **index of labor input** used by firm i, defined by

$$N_t(i) \equiv \left[\int_0^1 N_t(i,j)^{\frac{\varepsilon_W - 1}{\varepsilon_W}} dj \right]^{\frac{\varepsilon_W}{\varepsilon_W - 1}},$$

where

  - $N_t(i,j)$ is the quantity of type-j labor employed by firm i at date t,
  - ε_W is the (constant) elasticity of substitution between labor types,
  - $j \in [0,1]$ indexes the continuum of labor types.

Labor demand and wage index

- At each date t, given that each firm i employs an arbitrarily small fraction of each labor type j, it takes the nominal wages $[W(j)]_{j\in[0,1]}$ **as given**.

- The **intratemporal FOCs** of firms' optimization problem are similar to those of RH's optimization problem in Chapter 1, and lead to similar **demand schedules**:

$$N_t(i,j) = \left[\frac{W_t(j)}{W_t}\right]^{-\varepsilon_w} N_t(i)$$

for all $(i,j) \in [0,1]^2$, where

$$W_t \equiv \left[\int_0^1 W_t(j)^{1-\varepsilon_w} dj\right]^{\frac{1}{1-\varepsilon_w}}$$

is the **aggregate wage index**.

Intertemporal optimization problem

- In the same way as we got the aggregation result $\int_0^1 P_t(i)C_t(i)di = P_tC_t$ in Chapter 1, we get here that for all $i \in [0,1]$, $\int_0^1 W_t(j)N_t(i,j)dj = W_tN_t(i)$.

- Therefore, the **intertemporal optimization problem** of a price-resetting firm can be rewritten in exactly the same way as in Chapter 1.

- We assume here for simplicity that the elasticity of substitution between differentiated goods is constant over time, and we note it ε_p.

- We add superscript "p" to some of Chapter 1's notations, and thus note
  - μ_t^p the average (log) price markup at date t,
  - $\widehat{\mu}_t^p \equiv \mu_t^p - \mu^p = -\widehat{mc}_t$ the deviation of μ_t^p from its steady-state value,
  - θ_p the probability of not being allowed to reset one's price at a given date.

Price-inflation equation

- Therefore, the **intertemporal FOC** of firms' optimization problem can be rewritten, at the first order and in the neighborhood of the ZIRSS, as

$$\pi_t^p = \beta \mathbb{E}_t \left\{ \pi_{t+1}^p \right\} - \chi_p \widehat{\mu}_t^p,$$

where $\chi_p \equiv \frac{(1-\theta_p)(1-\beta\theta_p)}{\theta_p} \frac{1-\alpha}{1-\alpha+\alpha\varepsilon_p}$.

- This **price-inflation equation** can be interpreted as follows: whenever the current or expected future average price markups are below their desired value (which coincides with their steady-state value), firms currently resetting their prices raise the latter, thus generating positive inflation.

Utility function

- We consider a continuum of households indexed by $j \in [0,1]$.
- The **intertemporal utility function** of each household j at date 0 is

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t U \left[C_t(j), N_t(j) \right] \right\},$$

where

$$C_t(j) \equiv \left[\int_0^1 C_t(i,j)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} di \right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}}$$

and the **instantaneous utility function** U is the same as in Chapter 1.

Monopoly power

- We assume that each household supplies only one type of labor, and that each type of labor is supplied by only one household.
- This is why we index the continuum of households also by $j \in [0, 1]$.
- This implies that each household has some **monopoly power** in the labor market and is able to set its nominal wage (i.e., the price at which it supplies its specialized labor services).
- Alternatively, one may think of many households, with atomistic joint mass,
  - specializing in the same type of labor,
  - delegating their wage decision to a **trade union** acting in their interest.

Nominal-wage stickiness

- We model **nominal-wage stickiness** in the same way as price stickiness.
- So, at each date, only a fraction $1 - \theta_w$ of households, drawn randomly from the population, re-optimize their nominal wage, where $0 \leq \theta_w \leq 1$.
- We assume **full consumption-risk sharing** across households (through the means of a complete set of security markets).
- This implies that, at each date,
  - the marginal utility of consumption is equalized across households,
  - all the wage-resetting households choose the same wage, as they face the same problem (so that there is a **representative wage-resetting household**).

Wage-optimization problem

- At each date t, the representative wage-resetting household chooses W_t^* to maximize the expected discounted sum of instantaneous utilities generated over the (uncertain) period during which its wage will remain unchanged,

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta_w)^k U\left(C_{t+k|t}, N_{t+k|t}\right) \right\},$$

subject to the sequence of **labor-demand schedules** and **flow budget constraints** that are effective over this period, i.e., for $k \geq 0$,

$$N_{t+k|t} = \left(\frac{W_t^*}{W_{t+k}} \right)^{-\varepsilon_w} N_{t+k},$$

$$P_{t+k} C_{t+k|t} + \mathbb{E}_{t+k}\{Q_{t+k,t+k+1} D_{t+k+1|t}\} \leq D_{t+k|t} + W_t^* N_{t+k|t} - T_{t+k},$$

where the notations are defined on the next slide.

Notations

- $Q_{t,t+1}$ denotes the stochastic discount factor for one-period-ahead nominal payoffs at date t, common to all households.

- For households that last reoptimized their wage at date t, and for $k \geq 0$,
  - $C_{t+k|t}$ denotes consumption at date $t+k$,
  - $N_{t+k|t}$ denotes labor supply at date $t+k$,
  - $D_{t+k|t}$ denotes the (random) nominal payoff at date $t+k$ of the portfolio of securities bought at date $t+k-1$,
  - $\mathbb{E}_{t+k}\{Q_{t+k,t+k+1}D_{t+k+1|t}\}$ denotes therefore the market value at date $t+k$ of the portfolio of securities bought at date $t+k$.

- For $k \geq 0$, $N_{t+k} \equiv \int_0^1 N_{t+k}(i)di$ denotes aggregate employment at date $t+k$.

First-order condition I

- The **FOC** of this wage-optimization problem can be written as

$$\sum_{k=0}^{+\infty}(\beta\theta_w)^k \mathbb{E}_t \left\{ N_{t+k|t} \left[U_c\left(C_{t+k|t}, N_{t+k|t}\right) \frac{W_t^*}{P_{t+k}} + \mathcal{M}_w U_n\left(C_{t+k|t}, N_{t+k|t}\right) \right] \right\} = 0,$$

where $\mathcal{M}_w \equiv \frac{\varepsilon_w}{\varepsilon_w - 1}$, or equivalently

$$\sum_{k=0}^{+\infty}(\beta\theta_w)^k \mathbb{E}_t \left\{ N_{t+k|t} U_c\left(C_{t+k|t}, N_{t+k|t}\right) \left(\frac{W_t^*}{P_{t+k}} - \mathcal{M}_w MRS_{t+k|t} \right) \right\} = 0,$$

where $MRS_{t+k|t} \equiv -\frac{U_n\left(C_{t+k|t}, N_{t+k|t}\right)}{U_c\left(C_{t+k|t}, N_{t+k|t}\right)}$ is the marginal rate of substitution between consumption and work hours at date $t+k$ for households that last reset their wage at date t.

First-order condition II

- In the limit case of full wage flexibility ($\theta_w = 0$),

$$\frac{W_t^*}{P_t} = \frac{W_t}{P_t} = \mathcal{M}_w MRS_{t|t},$$

so that $\mathcal{M}_w$ is the wedge between the real wage and the marginal rate of substitution prevailing in the absence of wage rigidity, i.e. the **desired gross wage markup**.

- At the ZIRSS,

$$\frac{W^*}{P} = \frac{W}{P} = \mathcal{M}_w MRS.$$

Log-linearized FOC

- Therefore, log-linearizing the FOC around the ZIRSS yields the following **wage-setting rule**:

$$w_t^* = \mu^w + (1 - \beta\theta_w) \sum_{k=0}^{+\infty} (\beta\theta_w)^k \mathbb{E}_t \left\{ mrs_{t+k|t} + p_{t+k} \right\},$$

where $\mu^w \equiv \log \mathcal{M}_w$.

- The chosen wage w_t^* is thus increasing in
  - expected future prices, because households care about the purchasing power of their nominal wage,
  - expected future marginal disutilities of labor (in terms of goods), because households want to adjust their real wage accordingly, given expected future prices.

Individual and average MRS

- Given the assumptions of
  - complete asset markets,
  - separability between consumption utility and labor disutility,

  individual consumption is **independent of individual wage history**: for $k \geq 0$, $C_{t+k|t} = C_{t+k}$.

- Therefore, the (log) **individual MRS** can be written as

$$\begin{aligned} mrs_{t+k|t} &= \sigma c_{t+k|t} + \varphi n_{t+k|t} \\ &= \sigma c_{t+k} + \varphi n_{t+k|t} \\ &= mrs_{t+k} + \varphi(n_{t+k|t} - n_{t+k}) \\ &= mrs_{t+k} - \varepsilon_w \varphi(w_t^* - w_{t+k}), \end{aligned}$$

  where $mrs_{t+k} \equiv \sigma c_{t+k} + \varphi n_{t+k}$ is the (log) **average MRS**.

Rewriting the log-linearized FOC

- Therefore, the log-linearized FOC can be rewritten as

$$
\begin{aligned}
w_t^* &= \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\sum_{k=0}^{+\infty}(\beta\theta_w)^k\mathbb{E}_t\left\{\mu^w + mrs_{t+k} + \varepsilon_w\varphi w_{t+k} + p_{t+k}\right\} \\
&= \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\sum_{k=0}^{+\infty}(\beta\theta_w)^k\mathbb{E}_t\left\{(1+\varepsilon_w\varphi)w_{t+k} - \widehat{\mu}_{t+k}^w\right\} \\
&= \beta\theta_w\mathbb{E}_t\left\{w_{t+1}^*\right\} + (1-\beta\theta_w)\left[w_t - (1+\varepsilon_w\varphi)^{-1}\widehat{\mu}_t^w\right],
\end{aligned}
$$

where $\widehat{\mu}_t^w \equiv \mu_t^w - \mu^w$ denotes the deviation of the (log) average wage markup $\mu_t^w \equiv (w_t - p_t) - mrs_t$ from its steady-state level μ^w.

Wage-inflation equation I

- In the same way as the dynamics of the aggregate price index P_t in Chapter 1, the dynamics of the aggregate wage index W_t can be written as

$$W_t = \left[\theta_w \left(W_{t-1}\right)^{1-\varepsilon_w} + \left(1-\theta_w\right)\left(W_t^*\right)^{1-\varepsilon_w}\right]^{\frac{1}{1-\varepsilon_w}},$$

which can be log-linearized around the ZIRSS as

$$w_t = \theta_w w_{t-1} + \left(1-\theta_w\right) w_t^*.$$

- Therefore, the log-linearized FOC can be further rewritten as

$$\pi_t^w = \beta \mathbb{E}_t \left\{\pi_{t+1}^w\right\} - \chi_w \widehat{\mu}_t^w,$$

where $\pi_t^w \equiv w_t - w_{t-1}$ denotes wage inflation and $\chi_w \equiv \frac{(1-\theta_w)(1-\beta\theta_w)}{\theta_w(1+\varepsilon_w\varphi)}$.

- This **wage-inflation equation** is similar to the price-inflation equation.

Wage-inflation equation II

- This wage-inflation equation can be interpreted in a similar way as the price-inflation equation: when the average wage is below the level consistent with maintaining the desired markup, households readjusting their nominal wage will tend to increase the latter, thus generating positive wage inflation.

- This wage-inflation equation replaces the condition $w_t - p_t = mrs_t$ obtained in Chapter 1.

- The imperfect adjustment of nominal wages generates a **time-varying wedge between the real wage and the MRS** of each household, and, as a result, between the average real wage and the average MRS.

- This leads to variations in the average wage markup and, given the wage-inflation equation, also in wage inflation.

Euler equation

- Similarly as in Chapter 1, one FOC of households' optimization problem is the **Euler equation**

$$\frac{Q_t}{P_t} U_c(C_t, N_{t|t-k}) = \beta \mathbb{E}_t \left\{ \frac{U_c(C_{t+1}, N_{t+1|t-k})}{P_{t+1}} \right\}.$$

- This FOC equalizes, for a household that last reset its wage at date $t-k$,
  - the loss in utility resulting from the decrease in C_t required to purchase one bond at date t,
  - the gain in expected utility resulting from the increase in C_{t+1} entailed by the payoff of that bond at date $t+1$.

- The log-linearization of this Euler equation around the ZIRSS is

$$c_t = \mathbb{E}_t \{c_{t+1}\} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t \left\{ \pi^p_{t+1} \right\} - \bar{i} \right),$$

exactly like in Chapter 1.

Output gap

- Let y_t^n denote the **natural level of output**, i.e. the level of output in the **absence of nominal rigidities** (both price and wage rigidities).

- In the same way as in Chapter 1, y_t^n can be shown to be equal to

$$y_t^n = \vartheta_y^n + \psi_{ya}^n a_t,$$

where $\vartheta_y^n \equiv \frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha}\left[\log\left(\frac{1-\alpha}{1-\tau}\right) - \mu^p - \mu^w\right]$ and $\psi_{ya}^n \equiv \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}$.

- Let $\widetilde{y}_t \equiv y_t - y_t^n$ denote the **output gap**.

Real-wage gap

- Let ω_t^n denote the **natural real wage**, i.e. the real wage $\omega_t \equiv w_t - p_t$ in the **absence of nominal rigidities** (again, both price and wage rigidities).

- In the same way as in Chapter 1, ω_t^n can be shown to be equal to

$$\begin{aligned} \omega_t^n &= \log\left(\frac{1-\alpha}{1-\tau}\right) + (y_t^n - n_t^n) - \mu^p \\ &= \vartheta_w^n + \psi_{wa}^n a_t, \end{aligned}$$

  where $\vartheta_w^n \equiv \log\left(\frac{1-\alpha}{1-\tau}\right) - \frac{\alpha}{1-\alpha}\vartheta_y^n - \mu^p$, $\psi_{wa}^n \equiv \frac{1-\alpha\psi_{ya}^n}{1-\alpha}$, and n_t^n is work hours in the absence of nominal rigidities.

- Let $\widetilde{\omega}_t \equiv \omega_t - \omega_t^n$ denote the **real-wage gap**.

Rewriting the price-inflation equation

- Recall the price-inflation equation:

$$\pi_t^p = \beta \mathbb{E}_t \left\{ \pi_{t+1}^p \right\} - \chi_p \widehat{\mu}_t^p.$$

- Now, using the first-order approximation of the aggregate production function (implicitly established on Slide 40 below), we get, at the first order,

$$\begin{aligned} \widehat{\mu}_t^p \equiv \; & \mu_t^p - \mu^p = mpn_t - \log(1-\tau) - \omega_t - \mu^p = \log\left(\frac{1-\alpha}{1-\tau}\right) \\ & + y_t - n_t - \omega_t - \mu^p = \widetilde{y}_t - \widetilde{n}_t - \widetilde{\omega}_t \simeq -\frac{\alpha}{1-\alpha}\widetilde{y}_t - \widetilde{\omega}_t, \end{aligned}$$

where $\widetilde{n}_t \equiv n_t - n_t^n$ denotes the employment gap.

- Therefore, the **price-inflation equation** can be rewritten as

$$\pi_t^p = \beta \mathbb{E}_t \left\{ \pi_{t+1}^p \right\} + \kappa_p \widetilde{y}_t + \chi_p \widetilde{\omega}_t,$$

where $\kappa_p \equiv \frac{\alpha \chi_p}{1-\alpha}$.

Rewriting the wage-inflation equation

- Similarly, recall the wage-inflation equation:

$$\pi_t^w = \beta \mathbb{E}_t \left\{ \pi_{t+1}^w \right\} - \chi_w \widehat{\mu}_t^w.$$

- Now, at the first order,

$$\begin{aligned} \widehat{\mu}_t^w &\equiv \mu_t^w - \mu^w = \omega_t - mrs_t - \mu^w = \widetilde{\omega}_t - (\sigma \widetilde{y}_t + \varphi \widetilde{n}_t) \\ &\simeq \widetilde{\omega}_t - \left(\sigma + \frac{\varphi}{1-\alpha} \right) \widetilde{y}_t. \end{aligned}$$

- Therefore, the **wage-inflation equation** can be rewritten as

$$\pi_t^w = \beta \mathbb{E}_t \left\{ \pi_{t+1}^w \right\} + \kappa_w \widetilde{y}_t - \chi_w \widetilde{\omega}_t,$$

where $\kappa_w \equiv \left(\sigma + \frac{\varphi}{1-\alpha} \right) \chi_w$.

Other equilibrium conditions

- The price- and wage-inflation equations involve the endogenous variables π^p, π^w, $\widetilde{\omega}$, and $\widetilde{y}$, the first three of which are linked to each other through the **inflation identity**

$$\Delta\widetilde{\omega}_t = \pi_t^w - \pi_t^p - \Delta\omega_t^n.$$

- Using the goods-market-clearing condition $c_t = y_t$, the Euler equation can be rewritten as the same **IS equation** as in Chapter 1:

$$\widetilde{y}_t = \mathbb{E}_t\left\{\widetilde{y}_{t+1}\right\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left\{\pi_{t+1}^p\right\} - r_t^n\right),$$

where

$$r_t^n \equiv \bar{i} + \sigma\mathbb{E}_t\{\Delta y_{t+1}^n\} = \bar{i} + \sigma\psi_{ya}^n\mathbb{E}_t\{\Delta a_{t+1}\}$$

is the **natural rate of interest**.

List of equilibrium conditions

- Given $(a_t, i_t)_{t\in\mathbb{N}}$, $\left(\widetilde{y}_t, \widetilde{\omega}_t, \pi_t^p, \pi_t^w\right)_{t\in\mathbb{N}}$ is determined by
  - the IS equation $\widetilde{y}_t = \mathbb{E}_t\left\{\widetilde{y}_{t+1}\right\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left\{\pi_{t+1}^p\right\} - r_t^n\right)$,
  - the price-inflation equation $\pi_t^p = \beta\mathbb{E}_t\left\{\pi_{t+1}^p\right\} + \kappa_p\widetilde{y}_t + \chi_p\widetilde{\omega}_t$,
  - the wage-inflation equation $\pi_t^w = \beta\mathbb{E}_t\left\{\pi_{t+1}^w\right\} + \kappa_w\widetilde{y}_t - \chi_w\widetilde{\omega}_t$,
  - the inflation identity $\Delta\widetilde{\omega}_t = \pi_t^w - \pi_t^p - \Delta\omega_t^n$,

  for $t \in \mathbb{N}$.

- Given $\left(a_t, i_t, \widetilde{y}_t, \widetilde{\omega}_t, \pi_t^p, \pi_t^w\right)_{t\in\mathbb{N}}$, $(y_t, \omega_t, c_t, n_t)_{t\in\mathbb{N}}$ is determined by
  - the definitions $\widetilde{y}_t \equiv y_t - y_t^n$ and $\widetilde{\omega}_t \equiv \omega_t - \omega_t^n$,
  - the goods-market-clearing condition $c_t = y_t$,
  - the aggregate production function $y_t = (1-\alpha)n_t + a_t$,

  for $t \in \mathbb{N}$.

Determinacy condition for extended Taylor rules I

- Consider the following extension of **Taylor's** (1993) **rule**, noted R_1:

$$i_t = \bar{i} + \phi_p \pi_t^p + \phi_w \pi_t^w + \phi_y \widetilde{y}_t,$$

where $\phi_p \geq 0$, $\phi_w \geq 0$, and $\phi_y \geq 0$.

- Using this rule to replace i_t in the IS equation, we can rewrite the system made of the four structural equations (in their deterministic version) as $\mathbb{E}_t\{\mathrm{X}_{t+1}\} = \mathrm{A}_1 \mathrm{X}_t$, where

$$\mathrm{X}_t \equiv \begin{bmatrix} \widetilde{y}_t \\ \pi_t^p \\ \pi_t^w \\ \widetilde{\omega}_{t-1} \end{bmatrix} \text{ and } \mathrm{A}_1 \equiv \begin{bmatrix} 1 + \frac{\kappa_p}{\beta\sigma} + \frac{\phi_y}{\sigma} & \frac{\phi_p}{\sigma} - \frac{1+\chi_p}{\beta\sigma} & \frac{\phi_w}{\sigma} + \frac{\chi_p}{\beta\sigma} & \frac{\chi_p}{\beta\sigma} \\ \frac{-\kappa_p}{\beta} & \frac{1+\chi_p}{\beta} & \frac{-\chi_p}{\beta} & \frac{-\chi_p}{\beta} \\ \frac{-\kappa_w}{\beta} & \frac{-\chi_w}{\beta} & \frac{1+\chi_w}{\beta} & \frac{\chi_w}{\beta} \\ 0 & -1 & 1 & 1 \end{bmatrix},$$

so that R_1 ensures determinacy if and only if exactly three eigenvalues of A_1 are outside the unit circle (since the system has three non-predet. variables).

Determinacy condition for extended Taylor rules II

- As shown by Blasselle and Poissonnier (2016), this happens if and only if

$$\phi_p + \phi_w + \frac{1-\beta}{(1-\vartheta)\kappa_p + \vartheta\kappa_w}\phi_y > 1.$$

where $\vartheta \equiv \frac{\chi_p}{\chi_p+\chi_w}$.

- A 1-unit permanent increase in π^p leads to a 1-unit permanent increase in π^w (through the inflation identity) and, therefore, to a $\frac{1-\beta}{(1-\vartheta)\kappa_p+\vartheta\kappa_w}$-unit permanent increase in $\widetilde{y}$ (through the price- and wage-inflation equations).

- So the left-hand side of the **determinacy condition** above represents the permanent increase in the interest rate prescribed by R_1 in response to a 1-unit permanent increase in price inflation.

- Therefore, as in Chapter 3, the determinacy condition corresponds to the **Taylor principle**: in the long term, the (nominal) interest rate should rise by more than the increase in price inflation in order to ensure determinacy.

Social-planner allocation I

- Consider a **benevolent social planner** seeking to maximize RH's welfare given technology.
- Given the absence of state variable (such as the capital stock), its optimization problem is **static**: at each date t,

$$\underset{[C_t(i,j),N_t(i,j)]_{0\leq i\leq 1,0\leq j\leq 1}}{Max} \int_0^1 U\left[C_t(j), N_t(j)\right] dj$$

subject to

$$C_t(j) \equiv \left[\int_0^1 C_t(i,j)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} di\right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}} \text{ and } N_t(j) \equiv \int_0^1 N_t(i,j) di \text{ for } j \in [0,1],$$

$$C_t(i) = A_t N_t(i)^{1-\alpha} \text{ for } i \in [0,1],$$

$$C_t(i) \equiv \int_0^1 C_t(i,j) dj \text{ and } N_t(i) \equiv \left[\int_0^1 N_t(i,j)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dj\right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}} \text{ for } i \in [0,1].$$

Social-planner allocation II

- The **optimality conditions** are similar to their counterparts in Chapter 2:

$$C_t(i,j) = C_t(j) = C_t(i) = C_t \text{ for } i \in [0,1] \text{ and } j \in [0,1],$$
$$N_t(i,j) = N_t(j) = N_t(i) = N_t \text{ for } i \in [0,1] \text{ and } j \in [0,1],$$
$$-\frac{U_{n,t}}{U_{c,t}} = MPN_t,$$

  where $MPN_t \equiv (1-\alpha)A_t N_t^{-\alpha}$ is the average marginal product of labor.

- Similarly as in Chapter 2, the **first and second conditions** come from
  - the strict concavity of $C_t(j)$ in each $C_t(i,j)$ (when $\varepsilon_p < +\infty$),
  - the strict concavity of $N_t(i)$ in each $N_t(i,j)$ (when $\varepsilon_w < +\infty$),
  - the strict concavity of $C_t(i)$ in $N_t(i)$ (when $\alpha > 0$).

- As in Chapter 2, the **third condition** equalizes the MRS between consumption and work to the corresponding marginal rate of transformation.

Distortions

- The model is characterized by **four distortions**:
  1. monopolistic competition in the goods market,
  2. monopolistic competition in the labor market,
  3. sticky prices,
  4. sticky wages.

- The two **monopolistic-competition distortions** are effective
  - at the steady state (unless they are exactly offset by the subsidy τ),
  - not in response to shocks (given the absence of cost-push shocks).

- The two **nominal-rigidity distortions** are effective
  - in response to shocks (unless the desired price and wage are constant),
  - not at the steady state (since prices and wages are then constant).

Condition for natural-allocation efficiency I

- Consider the following value for the constant **employment subsidy** τ:

$$\tau = \frac{\mathcal{M}_p\mathcal{M}_w - 1}{\mathcal{M}_p\mathcal{M}_w},$$

  where

  - $\mathcal{M}_p \equiv \frac{\varepsilon_p}{\varepsilon_p - 1} > 1$ is the gross price markup under flexible prices,
  - $\mathcal{M}_w \equiv \frac{\varepsilon_w}{\varepsilon_w - 1} > 1$ is the gross wage markup under flexible wages.

- This value of τ exactly offsets the two monopolistic-competition distortions, i.e. removes the overall **steady-state distortion**.

- Therefore, it is such that the **natural allocation** (i.e. the flexible-price-and-wage equilibrium) is **efficient** (i.e. coincides with the social-planner alloc.).

Condition for natural-allocation efficiency II

- Indeed, if prices and wages were perfectly flexible, then
  - all firms would choose the same price at each date,
  - all households would choose the same wage at each date,

  so that the **first two optimality conditions** would be met.

- Moreover, these price P_t and wage W_t would be such that

$$\frac{W_t}{P_t} = -\frac{U_{n,t}}{U_{c,t}}\mathcal{M}_w \text{ and } P_t = \mathcal{M}_p \frac{(1-\tau)W_t}{MPN_t},$$

  so that the **third optimality condition** would be met when $\tau = \frac{\mathcal{M}_p\mathcal{M}_w - 1}{\mathcal{M}_p\mathcal{M}_w}$.

MP and the (efficient) natural allocation I

- In Chapter 2, in the absence of steady-state distortion and cost-push shocks,
  - the natural allocation was efficient,
  - MP could achieve the natural allocation (by setting $i_t = r_t^n$).

- Here, in the absence of steady-state distortion and cost-push shocks,
  - the natural allocation is also efficient, as we have just shown,
  - but **MP cannot achieve the natural allocation**, as we now show.

- The natural allocation requires that
  - $\widetilde{y}_t = 0$, so that output is at its natural level,
  - $\widetilde{\omega}_t = 0$, so that the real wage is at its natural level,
  - $\pi_t^p = 0$, so that all firms have the same price,
  - $\pi_t^w = 0$, so that all households have the same wage.

MP and the (efficient) natural allocation II

- Now, given $(a_t, i_t)_{t\in\mathbb{N}}$, $(\widetilde{y}_t, \widetilde{\omega}_t, \pi_t^p, \pi_t^w)_{t\in\mathbb{N}}$ is determined by
  - the IS equation $\widetilde{y}_t = \mathbb{E}_t\{\widetilde{y}_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}^p\} - r_t^n\right)$,
  - the price-inflation equation $\pi_t^p = \beta\mathbb{E}_t\{\pi_{t+1}^p\} + \kappa_p\widetilde{y}_t + \chi_p\widetilde{\omega}_t$,
  - the wage-inflation equation $\pi_t^w = \beta\mathbb{E}_t\{\pi_{t+1}^w\} + \kappa_w\widetilde{y}_t - \chi_w\widetilde{\omega}_t$,
  - the inflation identity $\Delta\widetilde{\omega}_t = \pi_t^w - \pi_t^p - \Delta\omega_t^n$.

- Therefore, whatever $(i_t)_{t\in\mathbb{N}}$, and in particular even for $(i_t)_{t\in\mathbb{N}} = (r_t^n)_{t\in\mathbb{N}}$, we cannot have $(\widetilde{y}_t, \widetilde{\omega}_t, \pi_t^p, \pi_t^w) = (0,0,0,0)$ for all $t \in \mathbb{N}$.

- Thus, MP cannot achieve the natural allocation: even if CB observes in real time the technology shock a_t (from which it can infer r_t^n), the **natural allocation is not feasible** (in the sense given to that term in Chapter 3).

- The reason is that to make the real wage coincide with the natural real wage, you need either flexible nominal wages, or flexible prices, or both.

Determination of the welfare-loss function I

- We now derive the second-order approximation of RH's utility around the ZIRSS.

- Recall from Chapter 2 that, for any variable Z_t, we have

$$\frac{Z_t - Z}{Z} \simeq \widehat{z}_t + \frac{\widehat{z}_t^2}{2},$$

where $\widehat{z}_t \equiv z_t - z$ is the log-deviation of Z_t from its ZIRSS value.

- Therefore, using the market-clearing condition $\widehat{c}_t = \widehat{y}_t$, we get

$$\int_0^1 [U_t(j) - U]\, dj \simeq U_c C \left(\widehat{y}_t + \frac{1-\sigma}{2} \widehat{y}_t^2 \right)$$
$$+ U_n N \left[\int_0^1 \widehat{n}_t(j) dj + \frac{1+\varphi}{2} \int_0^1 \widehat{n}_t(j)^2 dj \right].$$

Determination of the welfare-loss function II

- Up to a second-order approximation, we have

$$\widehat{n}_t + \frac{1}{2}\widehat{n}_t^2 \simeq \int_0^1 \widehat{n}_t(j)dj + \frac{1}{2}\int_0^1 \widehat{n}_t(j)^2 dj,$$

where $N_t \equiv \int_0^1 N_t(j)dj$ denotes aggregate employment at date t.

- Using the labor-demand equation $\widehat{n}_t(j) - \widehat{n}_t = -\varepsilon_w \widehat{w}_t(j)$, we also get

$$\begin{aligned}\int_0^1 \widehat{n}_t(j)^2 dj &= \int_0^1 \left[\widehat{n}_t(j) - \widehat{n}_t + \widehat{n}_t\right]^2 dj \\ &= \widehat{n}_t^2 - 2\widehat{n}_t\varepsilon_w \int_0^1 \widehat{w}_t(j)dj + \varepsilon_w^2 \int_0^1 \widehat{w}_t(j)^2 dj.\end{aligned}$$

- We admit the following result (whose proof is similar to Lemma 1's):

Lemma 3: *up to a second-order approx.,* $\int_0^1 \widehat{w}_t(j)dj \simeq \frac{\varepsilon_w - 1}{2} var_j\{w_t(j)\}$.

Determination of the welfare-loss function III

- We can then rewrite $\int_0^1 \left[U_t(j) - U\right] dj$ as

$$\int_0^1 \left[U_t(j) - U\right] dj \simeq U_c C \left(\widehat{y}_t + \frac{1-\sigma}{2} \widehat{y}_t^2 \right)$$
$$+ U_n N \left[\widehat{n}_t + \frac{1+\varphi}{2} \widehat{n}_t^2 + \frac{\varepsilon_w^2 \varphi}{2} var_j \{ w_t(j) \} \right].$$

- As in Chapter 2, we then derive a relationship between aggregate employment and output:

$$\begin{aligned} N_t &= \int_0^1 \int_0^1 N_t(i,j) dj di = \int_0^1 N_t(i) \int_0^1 \frac{N_t(i,j)}{N_t(i)} dj di = \Delta_{w,t} \int_0^1 N_t(i) di \\ &= \Delta_{w,t} \left(\frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}} \int_0^1 \left[\frac{Y_t(i)}{Y_t} \right]^{\frac{1}{1-\alpha}} di = \Delta_{w,t} \Delta_{p,t} \left(\frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}}, \end{aligned}$$

where $\Delta_{w,t} \equiv \int_0^1 \left[\frac{W_t(j)}{W_t} \right]^{-\varepsilon_w} dj$ and $\Delta_{p,t} \equiv \int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{\frac{-\varepsilon_p}{1-\alpha}} di$.

Determination of the welfare-loss function IV

- Therefore, we get (under the normalization $a = 0$)

$$(1-\alpha)\widehat{n}_t = \widehat{y}_t - a_t + d_{w,t} + d_{p,t},$$

where $d_{w,t} \equiv (1-\alpha)\log\Delta_{w,t}$ and $d_{p,t} \equiv (1-\alpha)\log\Delta_{p,t}$.

- We know from Lemma 1 that, up to a second-order approximation, $d_{p,t} \simeq \frac{\varepsilon_p}{2\Theta} var_i\{p_t(i)\}$, where $\Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon_p}$.

- We admit the following result (whose proof is also similar to Lemma 1's):

Lemma 4: *up to a second-order approx.,* $d_{w,t} \simeq \frac{(1-\alpha)\varepsilon_w}{2} var_j\{w_t(j)\}$.

Determination of the welfare-loss function V

- We can then rewrite $\int_0^1 [U_t(j) - U]\, dj$ as

$$\int_0^1 [U_t(j) - U]\, dj \simeq U_c C \left(\widehat{y}_t + \frac{1-\sigma}{2} \widehat{y}_t^2 \right) + \frac{U_n N}{1-\alpha} \left[\widehat{y}_t + \frac{\varepsilon_p}{2\Theta} var_i\{p_t(i)\} + \frac{\Upsilon}{2} var_j\{w_t(j)\} + \frac{1+\varphi}{2(1-\alpha)} \int_0^1 (\widehat{y}_t - a_t)^2\, dj \right] + t.i.p.,$$

where $\Upsilon \equiv (1-\alpha)(1+\varepsilon_w \varphi)\varepsilon_w$ and $t.i.p.$ stands again for "terms independent of policy."

- Let Φ denote the size of the steady-state distortion, implicitly defined by $-\frac{U_n}{U_c} = MPN(1-\Phi)$, and assumed to be "small" (i.e. a first-order term).

Determination of the welfare-loss function VI

- Using $MPN = (1-\alpha)\frac{Y}{N}$ and ignoring the $t.i.p.$ terms, we get

$$\begin{aligned}
\int_0^1 \frac{U_t(j)-U}{U_c C} dj &\simeq \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - (1-\Phi)\left[\widehat{y}_t + \frac{\varepsilon_p}{2\Theta} var_i\{p_t(i)\}\right. \\
&\quad \left. + \frac{\Upsilon}{2} var_j\{w_t(j)\} + \frac{1+\varphi}{2(1-\alpha)}(\widehat{y}_t - a_t)^2\right] \\
&\simeq \Phi\widehat{y}_t - \frac{1}{2}\left[\frac{\varepsilon_p}{\Theta} var_i\{p_t(i)\} + \Upsilon var_j\{w_t(j)\}\right. \\
&\quad \left. -(1-\sigma)\widehat{y}_t^2 + \frac{1+\varphi}{1-\alpha}(\widehat{y}_t - a_t)^2\right] \\
&= \Phi\widehat{y}_t - \frac{1}{2}\left[\frac{\varepsilon_p}{\Theta} var_i\{p_t(i)\} + \Upsilon var_j\{w_t(j)\}\right. \\
&\quad \left. + \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\widehat{y}_t^2 - 2\left(\frac{1+\varphi}{1-\alpha}\right)\widehat{y}_t a_t\right]
\end{aligned}$$

Determination of the welfare-loss function VII

$$= \quad \Phi\widehat{y}_t - \frac{1}{2}\left[\frac{\varepsilon_p}{\Theta} var_i\{p_t(i)\} + \Upsilon var_j\{w_t(j)\} + \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\left(\widehat{y}_t^2 - 2\widehat{y}_t\widehat{y}_t^e\right)\right]$$

$$= \quad \Phi\widehat{y}_t - \frac{1}{2}\left[\frac{\varepsilon_p}{\Theta} var_i\{p_t(i)\} + \Upsilon var_j\{w_t(j)\} + \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\left(\widetilde{y}_t\right)^2\right],$$

where we have used $\widehat{y}_t^e \equiv y_t^e - y^e = \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha} a_t$ and $\widetilde{y}_t \equiv y_t - y_t^n$ $= y_t - (y_t^e - y^e + y) = \widehat{y}_t - \widehat{y}_t^e$.

- As in Chapter 2, we get, up to first order, $\Phi \simeq \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) x^*$.

- Therefore, ignoring again the $t.i.p.$ terms, we get

$$\frac{U_t - U}{U_c C} \simeq -\frac{1}{2}\left[\frac{\varepsilon_p}{\Theta} var_i\{p_t(i)\} + \Upsilon var_j\{w_t(j)\} + \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\left(\widetilde{y}_t - x^*\right)^2\right].$$

Determination of the welfare-loss function VIII

- We admit the following result (whose proof is similar to Lemma 2's):

Lemma 5: $\sum_{t=0}^{+\infty} \beta^t var_j \{w_t(j)\} \simeq \frac{\theta_w}{(1-\beta\theta_w)(1-\theta_w)} \sum_{t=0}^{+\infty} \beta^t \left(\pi_t^w\right)^2$.

- Using Lemmas 2 and 5, we then get $\mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left(\frac{U_t - U}{U_c C} \right) \right\} \simeq t.i.p. - \frac{1}{2} \times$

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\frac{\varepsilon_p}{\chi_p} \left(\pi_t^p\right)^2 + \frac{(1-\alpha)\varepsilon_w}{\chi_w} \left(\pi_t^w\right)^2 + \left(\sigma + \frac{\varphi + \alpha}{1-\alpha} \right) \left(\widetilde{y}_t - x^*\right)^2 \right] \right\}.$$

- Hence the **welfare-loss function**

$$L_0 \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\lambda_p \left(\pi_t^p\right)^2 + \lambda_w \left(\pi_t^w\right)^2 + \lambda_y \left(\widetilde{y}_t - x^*\right)^2 \right] \right\},$$

where $\lambda_p \equiv \frac{\varepsilon_p}{\chi_p}$, $\lambda_w \equiv \frac{(1-\alpha)\varepsilon_w}{\chi_w}$, and $\lambda_y \equiv \left(\sigma + \frac{\varphi+\alpha}{1-\alpha} \right)$.

Interpretation of the welfare-loss function

- This **welfare-loss function** is identical to Chapter 2's (up to the constant multiplicative factor λ_p), except that it also involves π_t^w because
  - every variation in the general level of wages (i.e. every deviation of π_t^w from zero) implies a **wage dispersion**,
  - this wage dispersion is sub-optimal given the strict concavity of $N_t(i)$ in each $N_t(i,j)$ ($\varepsilon_w < +\infty$).

- The **weight** λ_w of the π_t^w-stabilization objective is increasing in
  - the elasticity of substitution between labor types ε_w,
  - the elasticity of output with respect to labor input $1-\alpha$,
  - the degree of wage stickiness θ_w,

  because these elasticities amplify the negative effect on aggregate productivity of any given wage dispersion, and θ_w raises the degree of wage dispersion resulting from any given wage-inflation rate different from zero.

Optimal MP

- We now study **optimal MP** in four alternative cases:
  1. sticky prices, flexible wages ($\theta_w \to 0$),
  2. flexible prices, sticky wages ($\theta_p \to 0$),
  3. sticky prices and wages (general case),
  4. sticky prices and wages (specific case $\kappa_p = \kappa_w$ and $\varepsilon_p = (1-\alpha)\varepsilon_w$).

- In all these cases, we assume that the employment subsidy exactly offsets the monopolistic-competition distortions:

$$\tau = \frac{\mathcal{M}_p \mathcal{M}_w - 1}{\mathcal{M}_p \mathcal{M}_w}.$$

- Therefore,
  - there is no steady-state distortion ($x^* = 0$),
  - the **natural allocation** ($\widetilde{y}_t = \widetilde{\omega}_t = 0$) **is efficient**.

Optimal MP when wages are flexible I

- **When $\theta_w \to 0$**, the model collapses to the **basic NK model** studied in Chapters 1 and 2 (in the absence of steady-state dist. and cost-push shocks).

- Indeed, when $\theta_w \to 0$, the **wage-inflation equation** becomes

$$\widetilde{\omega}_t = \left(\sigma + \frac{\varphi}{1-\alpha}\right)\widetilde{y}_t,$$

like in Chapter 1, where we had $\widetilde{\omega}_t \simeq \sigma\widetilde{c}_t + \varphi\widetilde{n}_t = \left(\sigma + \frac{\varphi}{1-\alpha}\right)\widetilde{y}_t$.

- Therefore, the **price-inflation equation** becomes

$$\pi_t^p = \beta\mathbb{E}_t\left\{\pi_{t+1}^p\right\} + \overline{\kappa}_p\widetilde{y}_t,$$

where $\overline{\kappa}_p \equiv \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\chi_p$, like in Chapter 1.

- The **IS equation** $\widetilde{y}_t = \mathbb{E}_t\left\{\widetilde{y}_{t+1}\right\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left\{\pi_{t+1}^p\right\} - r_t^n\right)$ and **identity** $\Delta\widetilde{\omega}_t = \pi_t^w - \pi_t^p - \Delta\omega_t^n$ remain unchanged, and both held in Chapter 1.

Optimal MP when wages are flexible II

- Finally, wage-inflation volatility becomes costless ($\lambda_w = 0$), so that the **welfare-loss function** simplifies to

$$L_0 \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\lambda_p \left(\pi_t^p \right)^2 + \lambda_y \left(\widetilde{y}_t \right)^2 \right] \right\},$$

like in Chapter 2 (up to the constant multiplicative factor λ_p).

- So we obtain the **same equilibrium conditions and welfare-loss function** as in Chapters 1 and 2 (without steady-state dist. and cost-push shocks).

- Therefore, given Chapter 2's results, **optimal MP**
  - **achieves the (efficient) natural allocation** ($\widetilde{y}_t = \widetilde{\omega}_t = 0$),
  - **tracks the natural rate of interest** ($i_t = r_t^n$),
  - **fully stabilizes price inflation** ($\pi_t^p = 0$),
  - lets wage inflation adjust as needed to make the real wage track the natural real wage ($\pi_t^w = \Delta\omega_t^n$).

Optimal MP when prices are flexible I

- **When $\theta_p \to 0$**, the **price-inflation equation** becomes

$$\widetilde{\omega}_t = \frac{-\alpha}{1-\alpha}\widetilde{y}_t.$$

- Therefore, the **wage-inflation equation** becomes

$$\pi_t^w = \beta \mathbb{E}_t \left\{ \pi_{t+1}^w \right\} + \overline{\kappa}_w \widetilde{y}_t,$$

where $\overline{\kappa}_w \equiv \left(\sigma + \frac{\varphi+\alpha}{1-\alpha} \right) \chi_w$.

- The **IS equation** and **inflation identity** remain unchanged:

$$\begin{aligned} \widetilde{y}_t &= \mathbb{E}_t \left\{ \widetilde{y}_{t+1} \right\} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t \left\{ \pi_{t+1}^p \right\} - r_t^n \right), \\ \Delta \widetilde{\omega}_t &= \pi_t^w - \pi_t^p - \Delta \omega_t^n. \end{aligned}$$

Optimal MP when prices are flexible II

- Finally, price-inflation volatility becomes costless ($\lambda_p = 0$), so that the **welfare-loss function** simplifies to

$$L_0 \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\lambda_w \left(\pi_t^w \right)^2 + \lambda_y \left(\widetilde{y}_t \right)^2 \right] \right\}.$$

- Optimal MP minimizes this welfare-loss function subject to the four equilibrium conditions on the previous slide.

- Therefore, **optimal MP**
  - **achieves the (efficient) natural allocation** ($\widetilde{y}_t = \widetilde{\omega}_t = 0$),
  - **fully stabilizes wage inflation** ($\pi_t^w = 0$),
  - lets price inflation adjust as needed to make the real wage track the natural real wage ($\pi_t^p = -\Delta\omega_t^n$).

Optimal MP in the general case I

- We now determine **optimal MP under commitment at date** 0 when both prices and wages are sticky ($\theta_p > 0$ and $\theta_w > 0$).

- As in Chapter 2, we proceed for simplicity as if CB, at each date t,
  - directly controlled not only i_t, but also $\widetilde{y}_t$, $\widetilde{\omega}_t$, π_t^p, and π_t^w,
  - observed the history of the exogenous shock $\left(a_{t-k}\right)_{k\geq 0}$.

- As in Chapter 2, since i_t appears only in the IS equation, we have

$$\min_{\left(i_t,\widetilde{y}_t,\widetilde{\omega}_t,\pi_t^p,\pi_t^w\right)_{t\in\mathbb{N}}} L_0 \text{ subject to (IS), (PI), (WI), (II)}$$

$$\Longleftrightarrow \min_{\left(\widetilde{y}_t,\widetilde{\omega}_t,\pi_t^p,\pi_t^w\right)_{t\in\mathbb{N}}} L_0 \text{ subject to (PI), (WI), (II),}$$

where (IS), (PI), (WI), and (II) denote respectively the IS equation, the price- and wage-inflation equations, and the inflation identity.

Optimal MP in the general case II

- The **reduced optimal-MP problem** is therefore, given $\widetilde{\omega}_{-1}$, to choose, at date 0, $\left(\widetilde{y}_t, \widetilde{\omega}_t, \pi_t^p, \pi_t^w\right)$ as a function of $(a_{t-k})_{0\leq k\leq t}$ for all $t \geq 0$, to minimize

$$L_0 \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\lambda_p \left(\pi_t^p\right)^2 + \lambda_w \left(\pi_t^w\right)^2 + \lambda_y (\widetilde{y}_t)^2 \right] \right\},$$

  subject to

  - the price-inflation equation $\pi_t^p = \beta \mathbb{E}_t \left\{ \pi_{t+1}^p \right\} + \kappa_p \widetilde{y}_t + \chi_p \widetilde{\omega}_t$ (PI),
  - the wage-inflation equation $\pi_t^w = \beta \mathbb{E}_t \left\{ \pi_{t+1}^w \right\} + \kappa_w \widetilde{y}_t - \chi_w \widetilde{\omega}_t$ (WI),
  - the inflation identity $\Delta \widetilde{\omega}_t = \pi_t^w - \pi_t^p - \Delta \omega_t^n$ (II),

  for all $t \geq 0$.

- Let $2\beta^t \xi_{1,t}$, $2\beta^t \xi_{2,t}$, and $2\beta^t \xi_{3,t}$ denote respectively the **Lagrange multipliers** associated with the constraints (PI), (WI), and (II) at date $t \in \mathbb{N}$.

Optimal MP in the general case III

- The corresponding **first-order conditions** (FOCs) are

$$\begin{aligned}
\lambda_y \widetilde{y}_t + \kappa_p \xi_{1,t} + \kappa_w \xi_{2,t} &= 0, \\
\lambda_p \pi_t^p - \Delta \xi_{1,t} + \xi_{3,t} &= 0, \\
\lambda_w \pi_t^w - \Delta \xi_{2,t} - \xi_{3,t} &= 0, \\
\chi_p \xi_{1,t} - \chi_w \xi_{2,t} + \xi_{3,t} - \beta \mathbb{E}_t \{\xi_{3,t+1}\} &= 0,
\end{aligned}$$

for $t \in \mathbb{N}$, where $\xi_{1,-1} \equiv 0$ and $\xi_{2,-1} \equiv 0$.

Optimal MP in the general case IV

- The system made of (PI), (WI), (II), and these FOCs can be written in Blanchard and Kahn's (1980) form $\mathbb{E}_t \left\{ \mathrm{Z}_{t+1} \right\} = \mathrm{A}_2 \mathrm{Z}_t + \mathrm{B} \Delta a_t$, where

$$\mathrm{Z}_t \equiv \left[\begin{array}{ccccccc} \widetilde{y}_t & \pi_t^p & \pi_t^w & \widetilde{\omega}_{t-1} & \xi_{1,t-1} & \xi_{2,t-1} & \xi_{3,t} \end{array} \right]',$$

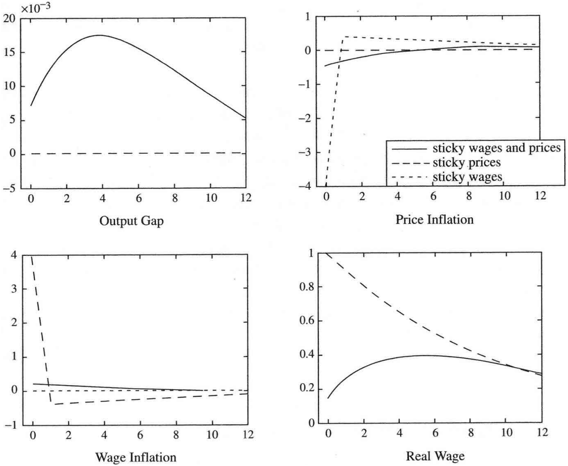
$\mathrm{A}_2 \in \mathbb{R}^{7 \times 7}$, and $\mathrm{B} \in \mathbb{R}^{7 \times 1}$.

- This system can be shown to meet Blanchard and Kahn's (1980) conditions and hence to have a unique stat. solution (Giannoni and Woodford, 2010).

- The next slide displays the responses of $\widetilde{y}_t$, π_t^p, π_t^w, and ω_t to ε_0^a at this unique local equilibrium, given the process $a_t = \rho_a a_{t-1} + \varepsilon_t^a$,
  - for sticky prices and sticky wages ($\theta_p > 0$ and $\theta_w > 0$),
  - for sticky prices and flexible wages ($\theta_p > 0$ and $\theta_w \rightarrow 0$),
  - for flexible prices and sticky wages ($\theta_p \rightarrow 0$ and $\theta_w > 0$).

Optimal MP in the general case V

Effects of a technology shock under optimal MP



Source: Galí (2015, C6).

Optimal MP in the general case VI

- As already seen, the natural allocation ($\widetilde{y}_t = 0$, $\omega_t = \omega_t^n$) can be achieved
  - when wages are flexible, by setting $\pi_t^p = 0$ and π_t^w such that $\omega_t = \omega_t^n$,
  - when prices are flexible, by setting $\pi_t^w = 0$ and π_t^p such that $\omega_t = \omega_t^n$.
- When both prices and wages are sticky, the natural allocation cannot be achieved, and **optimal MP strikes a balance** between
  - setting $(\widetilde{y}_t, \omega_t)$ as close as possible to $(0, \omega_t^n)$,
  - setting (π_t^p, π_t^w) as close as possible to $(0, 0)$.
- Therefore, in that case,
  - ω_t rises, but not as much as ω_t^n,
  - the fact that $\omega_t < \omega_t^n$ implies that $\widetilde{y}_t > 0$,
  - the rise in ω_t is obtained through a mix of lower π_t^p and higher π_t^w.

Optimal MP in a specific case I

- Lastly, we consider the case in which both prices and wages are sticky ($\theta_p > 0$ and $\theta_w > 0$), $\kappa_p = \kappa_w \equiv \kappa$, and $\varepsilon_p = (1-\alpha)\varepsilon_w \equiv \varepsilon$.

- In that case, the first three FOCs lead to

$$\chi_w \pi_t^p + \chi_p \pi_t^w = -\frac{\chi_p + \chi_w}{\varepsilon} \Delta \widetilde{y}_t \quad \text{for } t \geq 0, \quad \text{where } \widetilde{y}_{-1} \equiv 0.$$

- Let π_t denote a weighted average of price and wage inflation:

$$\pi_t \equiv (1-\vartheta)\pi_t^p + \vartheta \pi_t^w,$$

where, as a reminder, $\vartheta \equiv \frac{\chi_p}{\chi_p + \chi_w}$.

- The above optimality condition can then be rewritten as

$$\pi_t = -\frac{1}{\varepsilon} \Delta \widetilde{y}_t \quad \text{for } t \geq 0.$$

Optimal MP in a specific case II

- The previous optimality condition can be rewritten as

$$\widehat{q}_t = -\frac{1}{\varepsilon}\widetilde{y}_t \quad \text{for } t \geq 0,$$

where $\widehat{q}_t \equiv q_t - q_{-1}$ and $q_t \equiv (1-\vartheta)p_t + \vartheta w_t$ is a weighted average of the (log) price and wage levels.

- Now, the price- and wage-inflation equations can be combined to get

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa \widetilde{y}_t.$$

- The last two results, together with $\pi_t = \widehat{q}_t - \widehat{q}_{t-1}$, imply

$$\widehat{q}_t = \gamma \widehat{q}_{t-1} + \beta\gamma \mathbb{E}_t \{\widehat{q}_{t+1}\} \quad \text{for } t \geq 0,$$

where $\gamma \equiv \frac{1}{1+\beta+\kappa\varepsilon}$.

Optimal MP in a specific case III

- The last equation can be written in Blanchard and Kahn's (1980) form $\mathbb{E}_t \{Q_{t+1}\} = A_3 Q_t$, where

$$Q_t \equiv \begin{bmatrix} \widehat{q}_t \\ \widehat{q}_{t-1} \end{bmatrix} \text{ and } A_3 \equiv \begin{bmatrix} \frac{1}{\beta\gamma} & \frac{-1}{\beta} \\ 1 & 0 \end{bmatrix}.$$

- The eigenvalues of A_3,

$$\delta \equiv \frac{1-\sqrt{1-4\beta\gamma^2}}{2\beta\gamma},$$
$$\delta' \equiv \frac{1+\sqrt{1-4\beta\gamma^2}}{2\beta\gamma},$$

are such that $0 < \delta < 1$ and $\delta' > 1$.

Optimal MP in a specific case IV

- So the system has
  - one non-predetermined variable ($\mathbb{E}_t\{\widehat{q}_{t+1}\}$),
  - one eigenvalue outside the unit circle ($\delta' > 1$),

  and therefore a unique stationary solution.

- Given that $\widehat{q}_{-1} = 0$, this unique stationary solution is $\widehat{q}_t = 0$ for $t \geq 0$, which implies $\pi_t = 0$ and $\widetilde{y}_t = 0$ for $t \geq 0$.

- Therefore, **optimal MP fully stabilizes**
  - **a weighted average of price and wage inflation**, with the weight of price (wage) inflation increasing in the degree of price (wage) stickiness,
  - **the output gap**.